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LETTER TO THE EDITOR

# Chiral anomalies and the generalised index theorem

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**Abstract.** We show that the generalised Atiyah–Singer index theorem describes not only the U(1) anomaly, but also the non-Abelian anomaly. These anomalies are determined in arbitrary even dimensions without ambiguity of coefficients. The index theorem reveals new kinds of anomalies. The relationship between these anomalies and the topology of Yang–Mills fields is discussed.

In a Yang–Mills gauge field theory coupled to a massless chiral fermion, two different types of chiral anomalies are known. One is the anomalous divergence of global axial Noether current (U(1) anomaly), in four dimensions,

$$\begin{aligned}\partial_\mu J^\mu &= -\tfrac{1}{32}\pi^{-2}\,\mathrm{Tr}\,F^{\mu\nu}F^*_{\mu\nu}\\ &= -\tfrac{1}{8}\pi^{-2}\,\mathrm{Tr}\,\partial^\mu\varepsilon_{\mu\nu\rho\sigma}(A^\nu\,\partial^\rho A^\sigma+\tfrac{2}{3}A^\nu A^\rho A^\sigma)\end{aligned}\tag{1}$$

(metric $(++++)$, $\varepsilon_{1234}=1$, $A_\mu=A^a_\mu T^a$ where $T^a$ is the anti-Hermitian representation of the generator of the gauge group $G$), and the other is the anomalous covariant divergence of local gauge current (non-Abelian anomaly),

$$(D_\mu J^\mu)^a=\tfrac{1}{24}\pi^{-2}\,\mathrm{Tr}\,T^a\,\partial^\mu\varepsilon_{\mu\nu\rho\sigma}(A^\nu\,\partial^\rho A^\sigma+\tfrac{1}{2}A^\nu A^\rho A^\sigma).\tag{2}$$

Recently, Zumino *et al* (1984) and Stora (1983) found mathematical tools which gave these anomalies in arbitrary even dimensions. Here we briefly review their results.

First, the U(1) anomaly in $D=2n$ dimensions is given by

$$d^*j=\mathrm{ch}_n(E)=(1/n!)\,\mathrm{Tr}[-(\mathrm{i}/2\pi)F]^n,\tag{3}$$

where $\mathrm{ch}_n(E)$ is the $n$th Chern character (which is a $2n$ form) of the vector bundle $E$, on the fibre of which the Lie algebra of $G$ is represented, and the product implies the wedge product of differential forms. (For mathematical notation and details see Zumino *et al* (1984) and Eguchi *et al* (1980).)

In describing the non-Abelian anomaly, we use an elegant formalism by Stora (1983) introducing the Faddeev–Popov ghost $\omega$ and the BRS operator $S$ such that,

$$SA=-\mathrm{d}\omega-[A,\omega]m\qquad S\omega=-\tfrac{1}{2}[\omega,\omega].\tag{4}$$

$S$ is defined as an anti-derivative anti-commuting with the exterior derivative and odd differential forms. $\omega$ can be interpreted as a Mauer–Cartan form in the infinite-dimensional group of gauge transformations (Bonora and Cotta-Ramusino 1983).

The non-Abelian anomaly means that the fermionic effective action $\Gamma[A]$ is not BRS invariant (here the operator $S$ is equivalent to infinitesimal gauge transformation generated by $\omega$ because $\Gamma[A]$ does not include $\omega$),

$$S\Gamma[A]=G[\omega\,;A]. \tag{5}$$

It is important to notice that the fermionic determinant $\det \mathrm{i}\not{D}_A=\exp(-\Gamma[A])=\int \mathrm{D}\bar{\psi}\mathrm{D}\psi\,\exp(-\bar{\psi}\mathrm{i}\not{D}_A\psi)$ itself is not well-defined for a chiral fermion because the chiral Dirac operator $\mathrm{i}\not{D}$ acts on the spinor space $S^+$ of positive chirality while the image lies in the spinor space $S^-$ of negative chirality. So $\mathrm{i}\not{D}$ does not have an eigenvalue problem. This fact is closely related to the origin of the non-Abelian anomaly. However, the difference between the two effective actions $\Gamma[A']-\Gamma[A]=-\mathrm{Tr}\ln \mathrm{i}\not{D}_{A'}-(-\mathrm{Tr}\ln \mathrm{i}\not{D}_A)=-\mathrm{Tr}\ln \not{D}_A^{-1}\not{D}_{A'}=-\ln\det(\not{D}_A^{-1}\not{D}_{A'})$ is well defined because $\not{D}_A^{-1}\not{D}_{A'}$: $S^+\to S^+$ has eigenvalues. For this reason, (5) is well defined. Applying $S$ on (5) and noticing $S^2=0$, we get,

$$SG[\omega\,;A]=0. \tag{6}$$

This is the Wess–Zumino consistency condition for the non-Abelian anomaly (Wess and Zumino 1971). Stora gave a solution to this equation starting from the $(D+2)$-dimensional Chern character

$$\mathrm{ch}_{n+1}(E)=[1/(n+1)!](-\mathrm{i}F'/2\pi)^{n+1},$$

where $F'$ is given by

$$F'=DA'+\tfrac{1}{2}[A',A'],$$

with

$$A'=A+\omega, \qquad \mathrm{D}=\mathrm{d}+S.$$

The $(D+1)$-dimensional Chern–Simons secondary characteristic class $Q_{D+1}[A']$ with the property $\mathrm{ch}_{n+1}(E')=DQ_{D+1}[A']$ can be easily calculated (Zumino *et al* 1983, Stora 1983, Eguchi *et al* 1983). We expand it in powers of $\omega$:

$$Q_{D+1}[A']=Q^0_{D+1}[A]+Q^1_D[\omega\,;A]+Q^2_{D-1}[\omega\,;A]+\ldots+Q^{D+1}_0[\omega],$$

where $Q^q_p[\omega\,;A]$ is $p$th order in $A$ and $q$th order in $\omega$. It is easy to see that

$$G[\omega\,;A]=\int_{M_D} Q^1_D[\omega\,;A] \tag{7}$$

satisfies the condition (6). This gives the non-Abelian anomaly in $D=2n$ dimensions. Zumino *et al* (1983) also started from the $(D+2)$-dimensional Chern character to get the same result and gave a general formula for calculating (7).

Their derivation of non-Abelian anomaly is unsatisfactory for the following reasons. First, they used the Wess–Zumino condition (6) which is purely topological and its relation to the 'analytical side' $S\Gamma[A]$ was not considered. So it still requires perturbation theory to determine the coefficients. Second, it is not clear why one should start from the $(D+2)$-dimensional Chern character. What is the meaning of the additional two dimensions? Third, the relationship between the U(1) anomaly and the non-Abelian anomaly is not clear.

On the other hand, the relationship between U(1) anomaly and the Atiyah–Singer index theorem (Atiyah and Singer 1968a, b, Atiyah and Segal 1968) is well known

(Fujikawa 1979, 1980). The index theorem for a chiral Dirac operator is, on the $D=2n$-dimensional sphere $S^D$,

$$n_+ - n_- = \int_{S^D} \mathrm{ch}(E) = \int_{S^D} \mathrm{ch}_n(E) = \frac{1}{n!}\left(-\frac{\mathrm{i}}{2\pi}\right)^n \int_{S^D} \mathrm{Tr}\, F^n, \tag{8}$$

where $n_+$ and $n_-$ are the numbers of zero modes of $\mathrm{i}\not{D}$ and its adjoint $(\mathrm{i}\not{D})^*$, respectively, $\mathrm{ch}(E)=\mathrm{ch}_0(E)\oplus\mathrm{ch}_1(E)\oplus\ldots\oplus\mathrm{ch}_n(E)$ is the total Chern character and integration over $S^D$ picks up the $n$th Chern character. Equation (8) is, in essence, nothing other than the U(1) anomaly.

We shall generalise the index theorem (8) to the one which includes not only the U(1) anomaly but also the non-Abelian anomaly. We note that while the well known index theorem is the one for a single Dirac operator, we shall use the generalised index theorem for a family of Dirac operators (Atiyah and Singer 1971). In our approach, the unsatisfactory aspects of the methods by Stora (1983) and Zumino *et al* (1984) are remedied. First, notice that the index theorem combines the analytical side and the topological side. It really 'derives' anomalies without ambiguity of coefficients. Second, the meaning of the additional two dimensions (in the case of the non-Abelian anomaly) becomes clear. It is closely related to the topology of the Yang–Mills fields. Third, the index theorem gives a unified view of the topological nature of both U(1) and non-Abelian anomalies. Furthermore, it reveals new kinds of anomalies.

The topological origin of the non-Abelian anomaly has also been considered by Alvarez-Gaumé and Ginsparg (1983) and Gomez (1983). Their works and ours are overlapping and complementary.

Now we discuss the topology of the Yang–Mills gauge potential functional space. Let us first consider the four-dimensional Yang–Mills field. For definiteness, we take the gauge group $G=\mathrm{SU}(m)$ $(m\geqslant 3)$. The topological aspects of Yang–Mills fields have been well investigated by Atiyah and Jones (1978). The functional space of the gauge potential in Euclidian space $R^4$ is homotopically characterised by the asymptotic condition $A(x)\to \mathrm{U}^{-1}(x)\,\mathrm{dU}(x)$ (as $|x|\to\infty$). So the functional space may be identified with $\Omega^3(G)$, the space of maps from $S^3$ to $G$. Alternatively, one may consider the compactified space–time $S^4=R^4\cup\infty$. This time, the relevant functional space is $C(G)=A/g$, the space of potentials $(A)$ modulo four-dimensional gauge transformations $(g)$. $C(G)$ is again homotopically equivalent to $\Omega^3(G)$. We consider the topology of this space.

First, $\Pi_0(\Omega^3(G))=\Pi_3(G)=Z$, which means $\Omega^3(G)$ has connected components labelled by integer $k$, the well known instanton number. Next, $\Pi_2(\Omega^3(G))=\Pi_5(G)=Z$, that is, there exist 2-spheres in $\Omega^3(G)$ (or $C(G)$) that cannot be continuously deformed to one point. In fact, this topology will turn out to be related to the non-Abelian anomaly as we shall see later.

In general, $\Pi_{2n}(\Omega^3(G))=\Pi_{2n+3}(G)$. If we take $G$ such that $\Pi_{2n+3}(G)=Z$ for some $n\geqslant 2$, there exist non-trivial $2n$-spheres in $\Omega^3(G)$. This corresponds to new kinds of anomalies.

Now couple to the gauge field a massless chiral fermion in a complex representation of the gauge group $G$. This representation can be embedded in a representation of $\mathrm{U}(m)$ for some $m$.

The Bott periodicity theorem states that $\Pi_r(\mathrm{U}(m))=Z$ for odd $r<2m$. This topology is also important for the index problem of the chiral Dirac operator. For more details see Atiyah and Jones (1978).

We now illustrate the mathematical setting of the generalised index theorem.

On the analytical side, take $C(G)$ to be the base space and consider the 'index bundle' index $\not{D}$, which has an analytic connection $A^{(a)}$ on $C(G)$, $A^{(a)}$ is determined by the chiral Dirac operator $\not{D}_A$ parametrised by $A \in C(G)$†. The total Chern character of this bundle ch(index $\not{D}$) can be obtained from $A^{(a)}$ as usual.

On the topological side, take $C(G) \times S^4$ ($S^4$ being space–time) to be the base space. Define the topological connection $A^{(T)}$ on $C(G)$†. We choose the total connection of the topological bundle $E^{T}$ on $C(G) \times S^4$ as $\mathbb{A} = A^{(T)} + A$ where $A$ is the usual connection form (or gauge potential) on $S^4$. One obtains ch($E^{T}$) from $\mathbb{A}$. When one integrates ch($E^{T}$) on $S^4$ only, one gets another characteristic class on $C(G)$. The generalised index theorem (Atiyah and Singer 1971) states that this topological characteristic class coincides with the analytic one, i.e.

$$\mathrm{ch}(\mathrm{index}\ \not{D}) = \int_{S^4} \mathrm{ch}(E^{T}). \tag{9}$$

Recall that this is the direct sum of even differential forms on $C(G)$. On the right-hand side, one must pick up and integrate terms of 4-form on $S^4$.

Now let us look at the 0-form part of (9),

$$\mathrm{ch}_0(\mathrm{index}\ \not{D}) = \int_{S^4} \mathrm{ch}_2(E^{T}). \tag{10}$$

$\mathrm{ch}_0$(index $\not{D}$) is the 'dimension' of the bundle index $\not{D}$, which is equal to $n_+ - n_-$, and $\mathrm{ch}_2(E^{T}) = -\frac{1}{8}\pi^{-2}\,\mathrm{Tr}\,F^2$, so it coincides with (8) with $n = 2$. But this time the equation should be regarded as a function on $C(G)$. On the connected component labelled by instanton number $k$ it is constant, $n_+ - n_- = k$. In a word, the zero-form part of (9) describes the U(1) anomaly.

Next look at the 2-form part of (9), which reads,

$$\mathrm{ch}_1(\mathrm{index\ D}) = \int_{S^4} \mathrm{ch}_3(E^{T}). \tag{11}$$

To see that this describes the non-Abelian anomaly, we use a trick analogous to the inverse process of compactifying $R^4$ to $S^4$. Suppose a two-dimensional disk $D^2$ in $A$ on the boundary $S^1$ of which the connections are connected by one-parameter gauge transformations. If we project $A$ to $A/g = C(G)$, identifying gauge-equivalent connections, we get an $S^2$ in $C(G)$. We take the inverse process of this, that is, lift (11) from $S^2$ in $C(G)$ to $D^2$ in $A$. Since the Chern character can be written as the exterior derivative of the Chern–Simons secondary characteristic class, $\mathrm{ch}_1(\mathrm{index}\ \not{D}) = \delta Q^1[A^{(a)}]$, $\mathrm{ch}_3(E^{T}) = (\mathrm{d} + \delta) Q_5[A + A^{(T)}]$, where d and $\delta$ are exterior derivatives on $S^4$ and $A$, respectively. So, it is easy to see that on the gauge-equivalent boundary $S^1$ in $A$, (11) becomes,

$$Q^1[A^{(a)}] = \int_{S^4} Q_4^1[A^{(T)}; A], \tag{12}$$

where on the right-hand side we picked up terms of 4-form on $S^4$ from $Q_5$. Now we want to obtain an explicit expression of (12). We note that on the gauge-equivalent curve $S^1$, $\delta$ is equivalent to $S$ (BRS operator) and $A^{(a)} = -\not{D}^{-1} S \not{D}$, $A^{(T)} = \omega$ (1-form on

† The explicit forms of $A^{(a)}$ and $A^{(T)}$ are not relevant. But one may take $A^{(a)} = -\not{D}_A^{-1}\delta'\not{A}$ and $A^{(T)} = -\mathrm{D}_A^{-1}\delta' A$, where $\delta$ is the exterior derivative in $C(G)$ and $\mathrm{D}_A$ is the covariant derivative. (Notice $\delta'\not{D}_A = \delta'\not{A}$.)

generating a one-parameter gauge transformation on $S^1$). In general,

$$\begin{aligned}\mathrm{ch}_1(E) &= -(\mathrm{i}/2\pi)\,\mathrm{Tr}\,F = -(\mathrm{i}/2\pi)\,\mathrm{d}\,\mathrm{Tr}\,A,\\ \mathrm{ch}_3(E) &= (\mathrm{i}/48\pi^3)\,\mathrm{Tr}\,F^3\\ &= (\mathrm{i}/48\pi^3)\,\mathrm{dStr}(A(\mathrm{d}A)^2+\tfrac{3}{2}A^3\,\mathrm{d}A+\tfrac{3}{5}A^5)\end{aligned}$$

(Str: Symmetrised trace)

or

$$\begin{aligned}Q_1[A] &= -(\mathrm{i}/2\pi)\,\mathrm{Tr}\,A,\\ Q_5[A] &= (\mathrm{i}/48\pi^3)\,\mathrm{Str}[A(\mathrm{d}A)^2+\tfrac{3}{2}A^3\,\mathrm{d}A+\tfrac{3}{5}A^5].\end{aligned}$$

We replace $A$ and d in $Q_5[A]$ by $A+\omega$ and $\mathrm{d}+S$, and pick up terms of first order in $\omega$,

$$Q_4^1[\omega;A] = (\mathrm{i}/48\pi^3)\,\mathrm{Str}[\omega\mathrm{d}(A\,\mathrm{d}A+\tfrac{1}{2}A^3)].$$

So we get from (12),

$$\mathrm{i}/2\pi\,\mathrm{Tr}(\not{D}^{-1}S\not{D}) = \frac{\mathrm{i}}{48\pi^3}\int_{S^4}\mathrm{Str}[\omega\,d(A\,\mathrm{d}A+\tfrac{1}{2}A^3)], \tag{13}$$

where Tr of the left-hand side is the trace over functional space, $\gamma$ matrix and gauge group indices. The left-hand side can be rewritten as follows,†

$$(\mathrm{i}/2\pi)\,\mathrm{Tr}(\not{D}^{-1}S\not{D}) = -(1/2\pi\mathrm{i})\,\mathrm{Tr}(S\ln\mathrm{i}\not{D}) = -(1/2\pi\mathrm{i})S\ln(\det\mathrm{i}\not{D}) = (1/2\pi\mathrm{i})S\Gamma[A], \tag{14}$$

so that

$$S\Gamma[A] = -\frac{1}{24\pi^2}\int_{S^4}\mathrm{Str}[\omega\,\mathrm{d}(A\,\mathrm{d}A+\tfrac{1}{2}A^3)]. \tag{15}$$

This completes the derivation of the non-Abelian anomaly from the 2-form part of the generalised index theorem (9).

Now we briefly discuss the global integration of the non-Abelian anomaly. The Wess–Zumino consistency condition (6) means that the anomaly $G[\omega;A]$ can be locally integrated by infinitesimal gauge transformations. However, the possibility of global integration remains a question. Some authors (Ishikawa 1983, Rossi *et al* 1984) have discussed this problem and concluded that the coefficient should be an integer. In our approach, this corresponds to integrating 1-form (12) on $S^1$ in $A$, or equivalently, integrating 2-form (11) on the corresponding $S^2$ in $C(G)$. This automatically gives an integer, which assures the global integration in the exponential factor (i.e. $\exp(-\Gamma[A^{\mathrm{U}}])-\exp(-\Gamma[A])$ is a single-valued function of $\mathrm{U}(x)(\in\Omega^4(G))$ where $A^{\mathrm{U}}$ means the gauge transformation of $A$ by $\mathrm{U}(x)$).

The integer above depends on how to take $S^1$ in $A$ (or $S^2$ in $C(G)$). What is the meaning of this integer? On the analytical side, look at (14), which means phase change of fermionic determinant. (The modulus of det $\mathrm{i}\not{D}$ is of course gauge invariant.) When globally integrated, the integer equals the winding number of this phase change. On the topological side, notice that 1-parameter four-dimensional gauge transformations can be regarded as a map from $S^5$ to $G$. The integer indicates the homotopy class of this map, corresponding to $\Pi_5(G)=Z$.

† Even if $\not{D}$ has zero modes (i.e. $k\neq 0$), this quantity (14) is well defined.

Now we return to the index theorem (9) and discuss the meaning of the higher form parts. For example, we take the 4-form part,

$$\mathrm{ch}_2(\mathrm{index\ D}) = \int_{S^4} \mathrm{ch}_4(E^{\mathrm{T}}). \tag{16}$$

As before, we take $S^4$ in $C(G)$ and lift it to $D^4$ in $A$ on the boundary of which connections are connected by three-parameter gauge transformations. On this $S^3$, (16) becomes,

$$Q^3[A^{(\mathrm{a})}] = \int_{S^4} Q_4^3[A^{(\mathrm{T})}; A]. \tag{17}$$

We calculate this explicitly to get

$$-\frac{1}{24\pi^2}\mathrm{Tr}(\not{D}^{-1}S\not{D})^3 = \frac{1}{4!(2\pi)^4}\int_{S^4} \mathrm{Str}\{\omega^3[-\tfrac{3}{5}(\mathrm{d}A)^2 + \tfrac{4}{5}A^2\,\mathrm{d}A + \tfrac{2}{7}A^4]\}. \tag{18}$$

The right-hand side is non-trivial if $\Pi_7(G)$ is non-trivial. What is the meaning of this new anomaly? Notice the quantity on the left-hand side of (18) has the same form as $\mathrm{Tr}(U^{-1}\,\mathrm{d}U)^3$, which we use for evaluating the instanton number. In the above, we can regard $\not{D}$ as an element of infinite-dimensional Lie group and (18) means that there is an instanton-like configuration in the analytic bundle index $\not{D}$. More concretely, (18) means $\not{D}$ has $S^3$ phase factor and it is not invariant under three-parameter gauge transformations.

The physical meaning of this new 'anomaly' is not yet clear, but we propose a conjecture that this is the anomaly of hidden symmetries of the gauge theory.

The generalisation to arbitrary even dimensions is immediate. The index theorem reads,

$$\mathrm{ch}(\mathrm{index}\ \not{D}) = \int_{S^D} \mathrm{ch}(E^{\mathrm{T}}), \tag{19}$$

the zero form part of which is simply equation (8) as a function on $C(G)$ describing the U(1) anomaly. And the higher form part, when lifted to $A$ and converted to gauge-equivalent boundary $S^r$ ($r$: odd), becomes,

$$(\mathrm{i}/2\pi)^{s+1}(s!/r!)\,\mathrm{Tr}(\not{D}^{-1}S\not{D})^r = \int_{S^D} Q_D^r[\omega; A], \tag{20}$$

where $r = 2s+1$. In the special case of $r = 1$, this reduces to the non-Abelian anomaly,

$$S\Gamma[A] = 2\pi\mathrm{i}\int_{S^D} Q_D^3[\omega; A]. \tag{21}$$

The non-Abelian anomaly is closely related to the path-integral quantisation of the gauge field, though chiral anomalies are known to be 1-loop results. The crucial mathematical relation is $S^1 \wedge \Omega^4(G) = S^2 \wedge \Omega^3(G)$,† which means that one-parameter gauge transformations are equivalent to two-parameter physical variations of the gauge field. This is the origin of additional two dimensions, or 2-form in $\tilde{C}(G) = \Omega^3(G)$. Notice one wants to integrate fermionic effective action over $C(G)$ (physical gauge

† $\wedge$ means smash product. Take topological space $X$, $Y$ with base points $x_0, y_0$. $X \wedge Y$ is defined as $(X \times Y)/(X \vee Y)$ where $X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$.

potential space), not over $A$ (whole gauge potential space). Non-trivial characteristic classes of the generalised index theorem imply an obstruction to defining the action as a function over $C(G)$. The non-Abelian anomaly and Gribov ambiguity (Singer 1978) has the same mathematical origin: There exists no section in any non-trivial principal fibre bundle.

Now the difference between the U(1) anomaly and the other anomalies is clear. Because the U(1) anomaly has its origin in $\Pi_3(G)$, it is insensitive to four-dimensional gauge transformations. On the other hand, the non-Abelian anomaly is sensitive to one-parameter gauge transformations because it corresponds to $\Pi_5(G)$. In general, anomaly (20) is sensitive to $r$-parameter gauge transformations.

As we have seen, the chiral fermion can be used as a probe for investigating the topology of the Yang–Mills field. When one quantises the gauge field, one should keep in mind that there exist non-trivial cycles in $C(G)$.

The idea of the generalised index theorem can be applied to the gravitational anomalies in $(4n+2)$ dimensions discussed by Alvarez-Gaumé and Witten (1984). We shall discuss this in a future paper.

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*Note added.* After finishing this work, we were informed that Atiyah and Singer have also discussed the relationship of the generalised index theorem (Family's index theorem) to anomalies (Atiyah and Singer 1984).

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